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COAXIAL TURBULENT JETS

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Abstract—A theoretical treatment is described for the coaxial turbulent double jets produced when two miscible streams of fluid are emitted with different speeds in the same direction from an inner and a concentric outer nozzle into a region of fluid with which each of the streams is freely miscible. The treatment can be applied when the environment is at rest or is moving as a uniform laminar stream in the direction of the jet, but it does not include the effect of mixing across the outer boundary due to turbulence which may be present in the main stream. The theory, which is based on simple model assumptions about the mixing processes between the streams and with the environment, determines in principle the concentration of the two source fluids in each part of the double jet and the spread of the boundaries with increasing distance from the source. The final part of the solution is numerical in character; some representative cases are presented, and a general survey is given of the character of solutions for all nozzle conditions.

INTRODUCTION

THE main features of the turbulent jets produced in an extensive region of still fluid by the rapid release from a nozzle of any freely miscible fluid of approximately the same density are so familiar that they need not be described here (see, for example, Townsend [1]). When the ambient fluid is in uniform motion in the same direction as the jet, the behaviour may be expected to be basically similar provided that there is no turbulence in the main stream, and this case has been investigated elsewhere [2].

Although these simple jet flows are well understood, the analysis of the more complicated flow patterns that arise when two different but miscible streams are emitted from neighbouring sources is too complicated to be handled theoretically unless rather strongly simplified models are used. The experimental investigation of these double (or multiple) jets is also a good deal more difficult than that for a simple jet, and although some related problems have been discussed in the literature there seems to be little work that is directly relevant.

In spite of the difficulties associated with both theoretical and experimental investigations of these double jets, they remain one of the simplest cases of a wide range of engineering and geophysical problems that involve mixture between turbulent flows. Very little progress has been made with these problems, and it is useful to explore the possiblities of simple models whereever these may be available.

The purpose of this paper is to explore the possible use of the simple models which were developed to handle buoyant jets in a stratified environment by Morton *et al.* [3]. These depend primarily on the assumption that the ratio of the mean inflow velocity at an average edge of the jet to the mean velocity along its axis is constant at all stations of a given jet, and that the constant is the same for all jets provided that no substantial variations in density exist between the jet fluid and its environment. Although it is not immediately obvious how these simple ideas of mixing at the jet boundary should be extended to the more complicated structure of the coaxial multiple jet, it remains interesting to explore the consequences of at least one possible assumption. On this basis, definite predictions can be made and in due course may be checked experimentally as a test on the assumptions of the model. It will be shown that when the model is formulated mathematically, the resulting differential equations provide a simple problem of numerical solution, provided that a reasonably fast computing machine is available, so that the general behaviour of coaxial jets can be mapped out quickly. Two-dimensional double jets could be investigated in a similar way.

In the following treatment the density is taken as uniform throughout the flow, so that the solution corresponds to the case of *forced convection*. This should provide a reasonably good approximation up to reasonable distances from the source, in a region where the effects of inertia dominate these due to buoyancy. Although the treatment is described in terms of concentration, it applies equally to the forced convection of hot gas streams. At greater distances from the source it might be desirable to include the effects of buoyancy, and this can be done quite well, though with some increase in complexity.

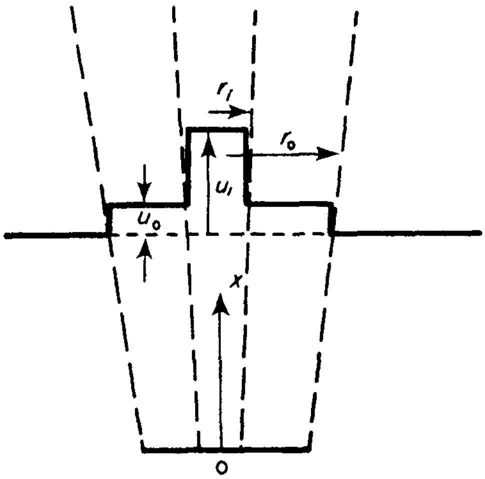
THE MODEL

The treatment can be applied to double jets in a still environment or in a uniform parallel stream, and the effects of buoyancy forces due to differences in density can also be taken into account, but the theory will actually be developed for the case of fluids of uniform density and a still environment, as this simplifies the working. The characteristics of the source can be described completely by specifying either (i) the momentum and mass flows from each nozzle, or (ii) the sectional area of each nozzle and the mean outflow speeds. The behaviour of jets can, in general, be understood best in terms of momentum flux and mass flux, so that the former specification of source conditions will be adopted. One of the simplifications in ordinary jet theories arises from the introduction of a "virtual point source", a hypothetical source which would give rise to the observed flow in the region above the actual source. However, in the present case there is no similar advantage to be gained, as it would be necessary to postulate two virtual sources at different positions on the axis of the double jet. Hence the origin of a cylindrical polar reference system will be taken at the centre of the inner nozzle, with the x-axis directed along the axis of the coaxial double jet. In most cases there will be some development length or region in which the flow in the inner and outer jets becomes fully turbulent, as in the case of an ordinary jet [4]; this can be compensated to a reasonable extent by an appropriate change of origin for any particular application, and need not be considered separately here.

In the developed region of turbulence there will be a central or "core" jet surrounded by an outer or "annular" jet. The supply of energy to the turbulence in the core flow arises from the difference in mean velocities of the core and annular flows, and the rate at which the core jet expands at the expense of surrounding fluid must depend on this velocity difference. Indeed it can depend very little on any other aspect of the system. The turbulence in the outer or annular flow arises essentially as a result of the outer shearing layer in contact with the ambient fluid, and hence the turbulence in the annular jet must be characterized by the difference between a representative mean velocity parallel to the axis within the annulus and the (parallel) velocity of the ambient fluid—which has been taken as zero for the time being. Thus in each case energy is supplied to the turbulence from the layer of relatively high velocity gradient which is towards the outer edge of the particular flow.

This picture of the double jet involves a considerable simplification and is likely to be poor in some regions, especially, for instance, where the outer flow is thin relative to its radius. However, it provides a mathematically tractable formulation, and it should at least provide a reasonable first approximation to double-jet behaviour. An advantage is that the assumptions are undisguised, and only one constant undetermined by the theory remains. In order to make practical use of these ideas on the double jet, it will be assumed that the jet profiles are similar for the core and the annular jets separately. There is then no further loss of physical information by assuming a uniform velocity across the core jet and a uniform, but in general different, velocity across the annular jet; these have, in the case of simple jet and plume, been called "top hat" profiles. The implications of these assumptions have been discussed further elsewhere [5].

Suppose that the average mean velocity in the core or inner jet is $u_i(x)$ and that in the annular or outer jet is $u_o(x)$, and that the mean radii of the inner and outer jets are $r_i(x)$ and $r_o(x)$, respectively. The ambient fluid has been taken at rest, but the general picture will be unchanged if it moves with uniform speed u parallel to Ox.



Equations which represent the conservation of mass and of momentum for an incompressible fluid (so that ρ is taken as constant and uniform throughout the field) can now be written for the inner jet as

$$\left.\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(r_i^2u_i) &= 2Er_i\,|u_i-u_o| - 2Er_iu_o,\\ \frac{\mathrm{d}}{\mathrm{d}x}(r_i^2u_i^2) &= 2Er_i\,|u_i-u_o|\,u_o - 2Er_iu_ou_i,\end{aligned}\right\} \quad (1)$$

where E is the entrainment constant defined, as in the case of a simple jet, as the ratio of the speed of inflow at a suitably defined outer edge (here the edge of the profile) to the mean speed of the jet fluid in the direction of the axis. Values have been given for E in the previous papers (e.g. [5]) and these should be applicable in this case without appreciable error. In each of these equations the first term on the right represents the effect of entrainment of fluid from the outer stream into the inner stream as the result of turbulence in the core jet; it may be noted that it is only the magnitude of the velocity difference which is significant. The second term represents the effect of mixing from the inner stream into the outer as the result of turbulence in the annular jet.

In the same way, the equations for the outer stream are

$$\left.\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}[(r_o^2-r_i^2)u_o]\\ &\quad= 2Er_ou_o - 2Er_i\,|u_i-u_o| + 2Er_iu_o,\\ &\frac{\mathrm{d}}{\mathrm{d}x}[(r_o^2-r_i^2)u_o^2]\\ &\quad= -\,2Er_i\,|u_i-u_o|\,u_o + 2Er_iu_ou_i,\end{aligned}\right\} \quad (2)$$

where the first, second and third terms on the right represent mixing of ambient fluid into the outer stream, of outer fluid into the inner stream, and of inner fluid into the outer stream, respectively.

The transported quantities which are most directly characteristic of the double jet are the mass and momentum fluxes in the inner and outer streams, and these can be represented by

$$v_i = r_i^2u_i,\quad v_o = (r_o^2-r_i^2)u_o,\quad m_i = r_i^2u_i^2,$$
$$m_o = (r_o^2-r_i^2)u_o^2,$$

where the suffix i is used for the inner stream and o for the outer one. The new forms of equations (1) and (2) in terms of these mass flux and momentum flux variables are

$$\frac{\mathrm{d}v_i}{\mathrm{d}x} = 2E\frac{v_i}{m_i^{1/2}}\left|\frac{m_i}{v_i}-\frac{m_o}{v_o}\right| - 2E\frac{v_i}{m_i^{1/2}}\frac{m_o}{v_o}, \quad (3)$$

$$\frac{\mathrm{d}v_o}{\mathrm{d}x} = 2E\sqrt{\left(\frac{v_o^2}{m_o}+\frac{v_i^2}{m_i}\right)}\frac{m_o}{v_o} - 2E\frac{v_i}{m_i^{1/2}}\left|\frac{m_i}{v_i}-\frac{m_o}{v_o}\right| + 2E\frac{v_i}{m_i^{1/2}}\frac{m_o}{v_o}, \quad (4)$$

$$\frac{\mathrm{d}m_i}{\mathrm{d}x} = 2E\frac{v_i}{m_i^{1/2}}\left|\frac{m_i}{v_i}-\frac{m_o}{v_o}\right|\frac{m_o}{v_o} - 2E\frac{v_i}{m_i^{1/2}}\frac{m_om_i}{v_ov_i}, \quad (5)$$

$$\frac{\mathrm{d}m_o}{\mathrm{d}x} = -\,2E\frac{v_i}{m_i^{1/2}}\left|\frac{m_i}{v_i}-\frac{m_o}{v_o}\right|\frac{m_o}{v_o} + 2E\frac{v_i}{m_i^{1/2}}\frac{m_om_i}{v_ov_i}. \quad (6)$$

It is apparent that these equations are likely to resist any attempt at direct integration, but they are in potentially suitable form for numerical integration. However, some initial reduction is necessary if a reasonable survey of the field of solutions is to be obtained with the minimum of labour.

It is clear from the physical character of the flow [and obvious from the addition of equations (5) and (6)] that the total flux of momentum with the double jet in an environment at rest is constant, i.e.

$$m_i + m_o = \text{const.}$$
$$= m_{is} + m_{os}, \qquad (7)$$

where m_{is}, m_{os} represent the momentum fluxes from the source carried in the inner and outer streams, respectively. Further boundary conditions are provided by specifying the mass fluxes at the source as proportional to v_{is} and v_{os}. (Note that the density ρ should strictly appear as a factor in a number of these terms, but it can be divided out of all the equations since it is a common factor of each term.) Equation (6) can now be replaced by relation (7), and, on substitution for m_o in equations (3), (4) and (5), and at the same time reduction to non-dimensional form under the transformations

$$v_i = v_{is}V_i, \quad v_o = v_{os}V_o, \quad m_i = m_{is}M_i,$$

$$x = \frac{v_{is}}{2Em_{is}^{1/2}}X,$$

these equations take the forms

$$\frac{\mathrm{d}V_i}{\mathrm{d}X} = M_i^{1/2}\left|1 - A\frac{V_i}{V_o}\frac{B-M_i}{M_i}\right| - AM_i^{1/2}\frac{V_i}{V_o}\frac{B-M_i}{M_o}, \qquad (9)$$

$$\frac{\mathrm{d}V_o}{\mathrm{d}X} = AM_i^{1/2}\sqrt{\left[\frac{B-M_i}{M_i}\left(1 + A^2\frac{V_i^2}{V_o^2}\frac{B-M_i}{M_i}\right)\right]} - AM_i^{1/2}\left|1 - A\frac{V_i}{V_o}\frac{B-M_i}{M_i}\right| + A^2M_i^{1/2}\frac{V_i}{V_0}\frac{B-M_i}{M_i}, \qquad (10)$$

$$\frac{\mathrm{d}M_i}{\mathrm{d}X} = A\frac{M_i^{3/2}}{V_i}\frac{V_i}{V_o}\frac{B-M_i}{M_i}\left|1 - A\frac{V_i}{V_o}\frac{B-M_i}{M_i}\right| - A\frac{M_i^{3/2}}{V_i}\frac{V_i}{V_o}\frac{B-M_i}{M_i}, \qquad (11)$$

where $A = v_{is}/v_{os}$ and $B = (m_{is} + m_{os})/m_{is}$ are the ratio of inner to outer mass flux from the source, and the ratio of total to inner momentum flux from the source, respectively. Also

$$M_o = m_o/m_{os} = (B - M_i)/(B-1). \qquad (12)$$

Although equations (9–11) are in some ways scarcely encouraging, it can nevertheless be seen immediately that *the double jet is represented completely in character though not in the scale by*

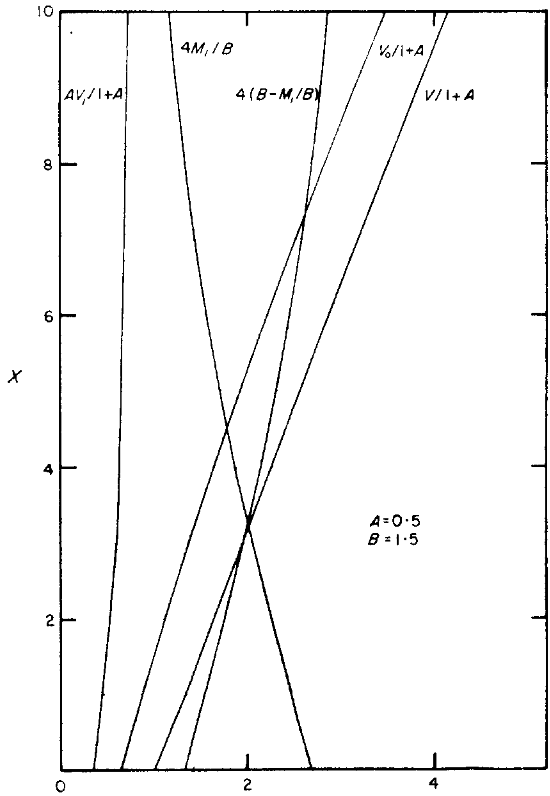


FIG. 1. The mass and momentum fluxes for inner and outer streams of the family of turbulent coaxial double jets ($A = 0{\cdot}5$, $B = 1{\cdot}5$) plotted against distance (X) from the common orifice non-dimensionally. The quantities plotted are: the total mass flow in the two streams $V/(1 + A)$, mass flow in the inner stream $AV_i/(1 + A)$, and mass flow in the outer stream $V_o/(1 + A)$, each in terms of total mass flow from the source; momentum flow in the inner stream M_i/B, and momentum flow in the outer stream $(B - M_i)/B$, each in terms of total momentum flow from the source. The total mass flow increases monotonically with distance, and there is a steady transfer of momentum from the inner to the outer stream.

the relative mass flux parameter A and the relative momentum flux parameter B. Thus, simultaneous variations in m_{is} and m_{os} (for example) which leave B unchanged and do not change v_{is} or v_{os} leave the character of the double jet unchanged except for a variation in the x-scale. It may be noted that the boundary conditions at $X = 0$ are $V_i = 1$, $V_0 = 1$, $M_i = 1$, and, incidentally, $M_o = 1$.

As an illustration, equations (9–11) have been solved numerically for the case $A = 0{\cdot}5$, $B = 1{\cdot}5$. These values correspond to $u_{is} = 4u_{os}$ and $r_{os} = 3r_{is}$, that is to an initial average core velocity 4 times the average velocity in the annular flow, and an outer annular orifice 8 times the cross-sectional area of the inner orifice. Fig. 1 shows in non-dimensional form a family of curves for the total mass flow [plotted as $V/(1 + A) = v_o/(v_{os} + v_{is})$], the inner mass flow $[AV_i/(1 + A) = v_i/(v_{os} + v_{is})]$, the outer mass flow $[V_o/(1 + A) = v_o/(v_{os} + v_{is})]$, the inner momentum flow $[M_i/B = m_i/(m_{os} + m_{is})]$, and the outer momentum flow $[(B - M_i)/B = m_o/(m_{os} + m_{is})]$, plotted against non-dimensional distance from the source $(X = 2Em_{is}^{1/2}x/v_{is})$; it may be noted that there is a steady transfer of momentum from the inner to the outer flow, and a steady increase in mass (or volume) flow in each of the inner and outer jets. This solution is further illustrated in Fig. (2a), where a family of curves for the inner radius (plotted in non-dimensional form as r_i/r_{is}), the outer radius (r_o/r_{is}), the inner velocity (u_i/u_{is}), and the outer velocity (u_o/u_{is}) are plotted against non-dimensional distance from the source $(2Em_{is}^{1/2}x/v_{is})$. Note that the same scale length r_{is} and the same scale velocity u_{is} are used for each jet in order that the figure will give an immediate picture of the double jet.

CONCENTRATIONS OF INNER AND OUTER SOURCE FLUIDS

It is obviously of interest to follow the concentrations of the two fluids emitted from the inner and outer source, respectively, through the various stages of the double jet, even though they may sometimes have no special properties by which they can be distinguished. In an actual flow, fluid brought into the edge of a jet will take a certain time to spread across it, and profiles of concentration (like profiles of velocity) will be roughly Gaussian in shape. However, for the present model it will again be sufficient to use mean values and hence to take a profile across the inner jet with uniform concentration $C_{1i}(x)$ of the fluid 1 emitted from the inner source, and with uniform concentration $C_{1o}(x)$ of fluid 1 across the outer jet, where C measures the proportion by volume of source fluid in the mixture. Similarly the concentration of fluid 2 emitted from the outer source can be taken as $C_{2i}(x)$ across the inner and $C_{2o}(x)$ across the outer jets. The conditions at the source, $x = 0$, are then:

$$C_{1i}(0) = 1, \quad C_{1o}(0) = 0; \quad C_{2i}(0) = 0, \quad C_{2o}(0) = 1;$$

and C_1, C_2 are zero throughout the ambient fluid.

The equations representing conservation of volume (or mass, as density has been taken as uniform) for fluid 1 from the inner source can now be written for the inner and outer jets respectively as

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(v_i C_{1i}) &= 2Er_i|u_i - u_o|C_{1o} - 2Er_i u_o C_{1i}, \\ \frac{\mathrm{d}}{\mathrm{d}x}(v_o C_{1o}) &= -2Er_i|u_i - u_o|C_{1o} \\ &\quad + 2Er_i u_o C_{1i}. \end{aligned}\right\} \quad (13)$$

The corresponding equations for fluid 2 emitted from the outer source are

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(v_i C_{2i}) &= 2Er_i|u_i - u_o|C_{2o} - 2Er_i u_o C_{2i}, \\ \frac{\mathrm{d}}{\mathrm{d}x}(v_o C_{2o}) &= -2Er_i|u_i - u_o|C_{2o} \\ &\quad + 2Er_i u_o C_{2i}. \end{aligned}\right\} \quad (14)$$

The pairs of equations (13) and (14) reduce under the transformations introduced above (and substituting from earlier equations) to the non-dimensional forms

$$\frac{\mathrm{d}C_{1i}}{\mathrm{d}X} = M_i^{1/2}\left|1 - A\,\frac{V_i}{V_o}\frac{B - M}{M_i}\right|\frac{C_{1o} - C_{1i}}{V_i}, \quad (15)$$

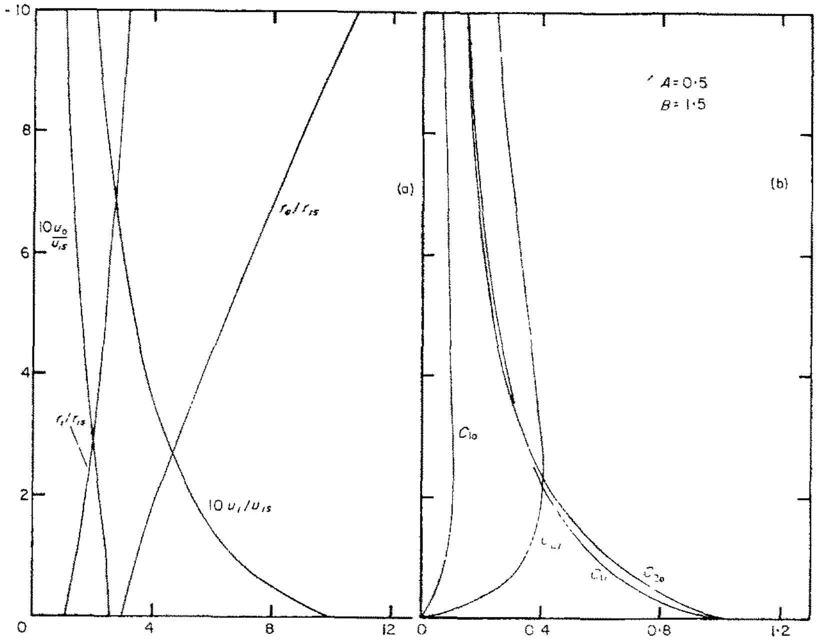


FIG. 2. The behaviour of the family of double jets $A = 0{\cdot}5$, $B = 1{\cdot}5$. (a) Non-dimensional curves showing the variation with distance (X) from the source of the radii of the inner (r_i/r_{is}) and outer (r_o/r_{is}) streams, and of the mean velocities of the inner (u_i/u_{is}) and outer (u_o/u_{is}) streams. The same reference radius (r_{is} which corresponds to the "effective radius" of the inner stream at the source) and reference velocity (u_{is}, the "effective mean velocity" of the inner stream at the source) are used for each stream so that direct comparison is possible. (b) Non-dimensional curves showing the variation of concentration (defined as proportion by volume) of: fluid in the inner stream which has come from the inner orifice (C_{1i}), fluid from the inner orifice in the outer stream (C_{1o}), fluid from the outer orifice in the inner stream (C_{2i}), and fluid from the outer orifice in the outer stream (C_{2o}). Note that C_{2i} soon exceeds C_{2o}, but that C_{1i} remains larger than C_{1o}.

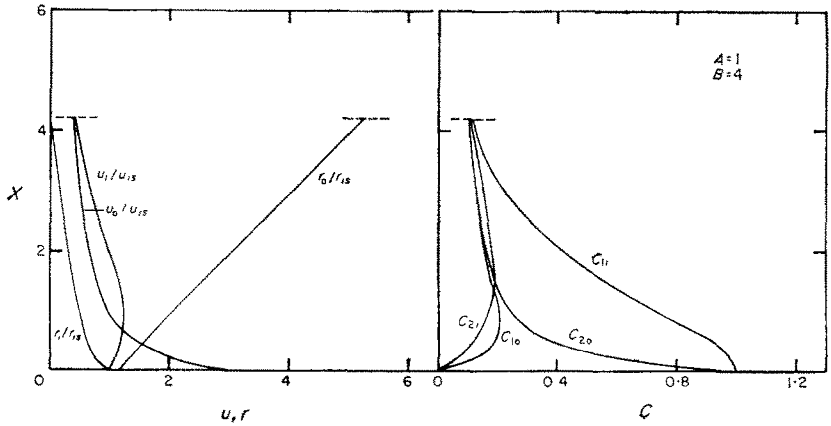


FIG. 3. Curves showing the variation with distance of the radii of inner and outer streams (r_i/r_{is} and r_o/r_{is}), of the mean velocities of inner and outer streams (u_i/u_{is} and u_o/u_{is}), and of the various concentrations C of source fluids in the two streams, plotted non-dimensionally for the family of turbulent double jets ($A = 1$, $B = 4$). This is a case in which the inner stream is completely absorbed by the outer stream.

$$\frac{dC_{2i}}{dX} = M_i^{1/2}\left|1 - A\frac{V_i}{V_o}\frac{B - M_i}{M_i}\right|\frac{C_{2o} - C_{2i}}{V_i}, \quad (16)$$

$$\frac{dC_{1o}}{dX} = - AM_i^{1/2}\sqrt{\left[\frac{B - M_i}{M_i}\left(1 + A^2\frac{V_i^2}{V_o^2}\frac{B - M_i}{M_i}\right)\right]}\frac{C_{1o}}{V_o} + A^2 M_i^{1/2}\frac{V_i}{V_o}\frac{B - M_i}{M_i}\frac{C_{1i} - C_{1o}}{V_o}, \quad (17)$$

$$\frac{dC_{2o}}{dX} = - AM_i^{1/2}\sqrt{\left[\frac{B - M_i}{M_i}\left(1 + A^2\frac{V_i^2}{V_o^2}\frac{B - M_i}{M_i}\right)\right]}\frac{C_{2o}}{V_o} + A^2 M_i^{1/2}\frac{V_i}{V_o}\frac{B - M_i}{M_i}\frac{C_{2i} - C_{2o}}{V_o}; \quad (18)$$

subject to the boundary conditions at $X = 0$

$$C_{1i} = 1, \quad C_{1o} = 0, \quad C_{2i} = 0, \quad C_{2o} = 1.$$

This system of equations (15–19) can readily be solved in the same routine as equations (9–11). As an illustration, the solution for $A = 0{\cdot}5$, $B = 1{\cdot}5$ has been continued and the family of concentration curves is shown in Fig. 2(b). The amount of ambient fluid which has been mixed into either the inner or the outer jet can be found at any height as the proportion by volume $1 - C_{1i} - C_{2i}$ or $1 - C_{1o} - C_{2o}$ for the inner and outer jets, respectively.

SURVEY OF THE SOLUTIONS

A typical solution of simple type has already been described and illustrated in Figs. 1 and 2; this was for $A = 0{\cdot}5$, $B = 1{\cdot}5$, to correspond with sources in which the area of the outer or annular orifice is 8 times that of the inner orifice and the inner mean velocity is 4 times the outer mean velocity. Both inner and outer jets increase steadily in cross section with increasing distance from the source, at least up to the level $X = 10$.

A second type of solution is illustrated in Fig. 3; this is for the case $A = 1$, $B = 4$, corresponding to $u_{os} = 3u_{is}$ and $r_{os} = (4/3)^{1/2}r_{is}$. In this case the thin but vigorous annular jet steadily encroaches on the slower inner jet, consuming it entirely by the height of $X = 4{\cdot}4$. It is easy to be too casual in referring to "inner" and "outer" jets, and it must always be remembered that such a description is meaningful only so long as the two can be distinguished by some possible physical measurement. In this case the inner jet vanishes a little above $X = 4{\cdot}4$; however, below this level there is a considerable difference in concentration of fluid issuing from the inner nozzle over almost the whole height, and this provides a valid reason for use of inner and outer jets. In contrast, the concentration of fluid emitted from the outer orifice is sensibly uniform over the whole jet by a height of $X = 1$.

An alternative type of "cut-off" solution might be anticipated in cases where a very vigorous inner jet is enclosed by a thin, slow outer jet, when it might be supposed that the outer stream would be engulfed by the inner stream. In fact this does not appear to be a possible solution to the set of equations which has been developed, and, although the outer stream may at first suffer a net loss of fluid and a contraction of cross-section, this trend is soon reversed because of the net momentum transport from inner to outer streams. Such a case is illustrated in Fig. 4, which shows the solution corresponding to $A = 5$, $B = 1{\cdot}01$, with $u_{is} = 20u_{os}$ and $r_{os} = \sqrt{5}r_{is}$. The outer diameter of the jet decreases sharply at first, but at the same time there is appreciable transport of momentum to the outer stream so that this effect is soon reversed. It may be noted that the concentration of fluid from the outer orifice in the inner jet (C_{2i}) soon exceeds that in the outer jet, where there is continual dilution by loss to the inner stream, by entrainment of ambient fluid and by entrainment of fluid from the inner orifice out of the inner stream.

While it is interesting to identify the possible solution types, a much more useful procedure would be to survey the solutions generally. Although the actual scale of the double jet will depend in detail on conditions at its sources, a general survey of these double jets can be carried out by finding the dependence on A and B alone. Moreover, if the distribution of the jet in space is of secondary concern, the independent variable X can be eliminated entirely (cf. [2]), and equations (9–11) reduced to a pair in V_o, V_i and M_i only, or in variables formed from

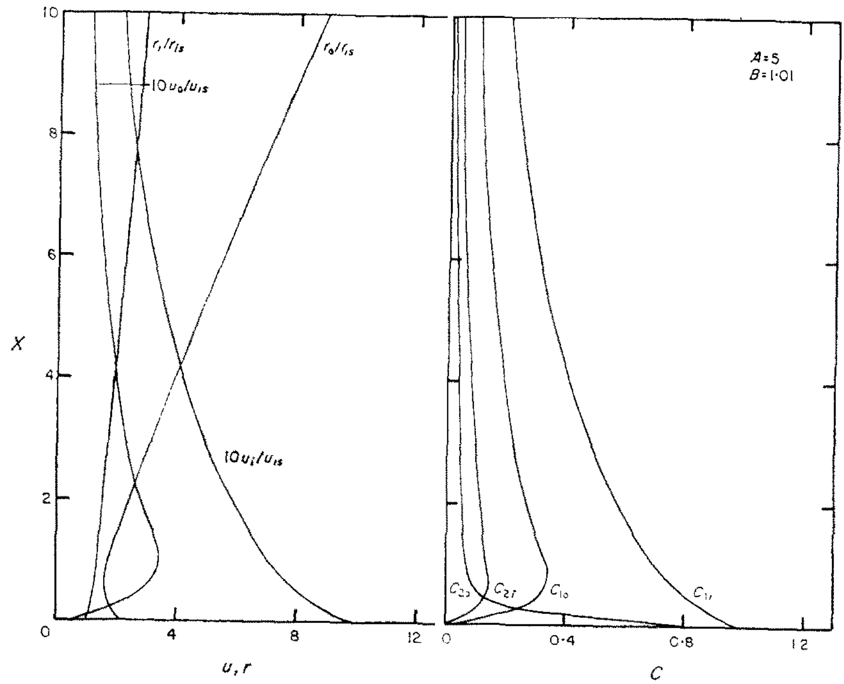


FIG. 4. A set of curves showing the variation with distance of radius, mean velocity and concentration for the double-jet family ($A = 5$, $B = 1{\cdot}01$). This is a case in which the outer stream is at first eroded rapidly, but then recovers and subsequently spreads vigorously itself.

combinations of these. The new independent variable must have the property of monotone variation with X (which will now remain unknown for the time being), and as none of these variables need have this behaviour separately, it will be taken as

$$V = V_o + AV_i = \frac{v_{os}V_o + v_{is}V_i}{v_{os}} = \frac{v_o + v_i}{v_{os}} = (1 + A)\frac{v_o + v_i}{v_{os} + v_{is}},$$

which is proportional to the total mass flux in the combined jets. The new variable $V_o + AV_i$ is certainly monotone increasing with X, as there can be only *entrainment* (or mixing *in*) at the outer edge of the double jet; in contrast, the simplest total momentum $m_i + m_o$ is of course constant.

The reduced equations are

$$\frac{\mathrm{d}V_i}{\mathrm{d}V} = \frac{\left|1 - \dfrac{AV_i}{V - AV_i}\dfrac{B - M_i}{M_i}\right| - \dfrac{AV_i}{V - AV_i}\dfrac{B - M_i}{M_i}}{A\left(\dfrac{B - M_i}{M_i}\right)^{1/2}\left(1 + \dfrac{A^2V_i^2}{(V - AV_i)^2}\dfrac{B - M_i}{M_i}\right)^{1/2}}, \tag{20}$$

$$\frac{\mathrm{d}M_i}{\mathrm{d}V} = \frac{B - M_i}{V - AV_i} \times \frac{\left|1 - \dfrac{AV_i}{V - AV_i}\dfrac{B - M_i}{M_i}\right| - 1}{\left(\dfrac{B - M_i}{M_i}\right)^{1/2}\left(1 + \dfrac{A^2V_i^2}{(V - AV_i)^2}\dfrac{B - M_i}{M_i}\right)^{1/2}}, \tag{21}$$

and in addition $V_o = V - AV_i$ and

$$(B - 1)\,M_o = B - M_i.$$

The corresponding boundary conditions at the source $V = 1 + A$ are that $V_i = 1$ and $M_i = 1$.

If much detail is wanted in the solution for a particular case, it will of course be necessary to solve the set of equations (9–11, 15–18) as has been done above. On the other hand, the main point of interest may be which of the solution types illustrated above is to be expected in practise for specified values of the parameters, and this information can be found from a survey of the solutions using equations (20) and (21). Numerical solutions for these equations can be obtained very rapidly using an electronic computer of reasonable speed.

The results of a rather restricted survey of this kind are illustrated in Fig. 5. This survey was based on the sole factor of "cut-off" or "extinguishment" of the inner jet, and the solution programme was arranged to give for each pair of values (A, B) the "height"—measured in terms of the total mass flux parameter $V/(1 + A)$—at which cut-off occurred, provided that this happened for $V/(1 + A) \leqslant 20$. The curves of Fig. 5 represent contours in the (A, B)—plane on which cut-off occurs at the marked values of $V/(1 + A)$. [Note that the variable $V/(1 + A)$ has been used because $V = 1 + A$ at the source of the double jet.] Thus, for example, each member of the family of double jets having $A = 1{\cdot}6$ and $B = 1{\cdot}8$ will have an inner jet which tapers to a point at the level where the total mass flux is represented by $V/(1 + A) = 10$, or $V = 26$. It is clear that all double jets represented by (A, B) values lying above the lowest contour will experience cut-off of the inner jet at appropriate V levels. But what about double jets of the region between this contour and the axis $B = 1$ (the value of B cannot be less than one)? There will obviously be cut-off of the inner jet at higher levels in terms of $V/(1 + A)$ for points close below the lowest contour marked. Points close to the A-axis correspond to cases in which most of the momentum is released with the inner stream from the source, and if A is large also then most of the volume release is from the inner orifice. These cases with strong inner jets surrounded by weak outer jets offer the greatest likelihood of cut-off of the outer stream. However, the survey uncovered no case in which the inner stream absorbed the outer

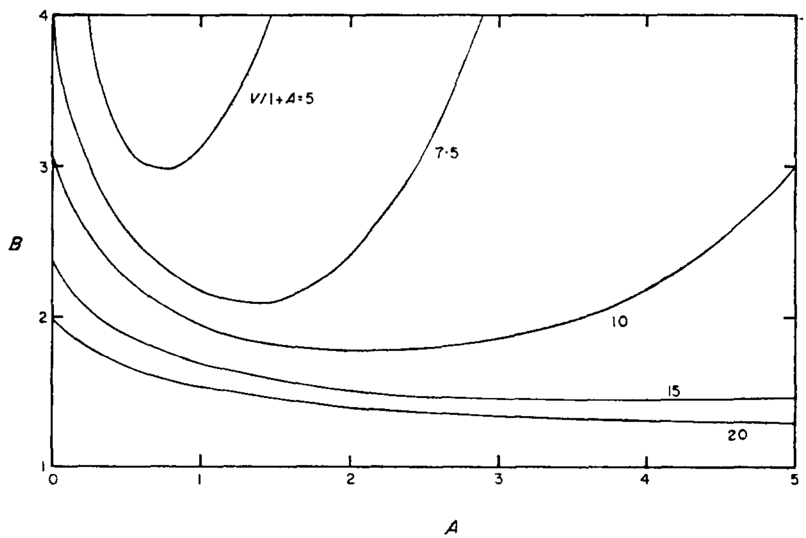


FIG. 5. The results of a survey of the behaviour of double jets based on the relationship between the values (A, B) and the distance from the source to cut-off of the inner jet. The curves are contours connecting (A, B) points at which the "distance" to cut-off has the same value, measured in terms of the total volume flux divided by the total volume flux from the source. Thus, all double jets corresponding with (A, B) values on the contour $V/(1 + A) = 10$ will exhibit cut-off of the inner jet at the level where the total volume flux is 10 times that from the source.

stream, even though it was carried to the point $A = 500$, $B = 1{\cdot}01$. The rate of momentum transfer to the outer stream is apparently sufficient to establish a growing outer stream in all the cases which were considered.

The results of the survey show that the outer stream is never absorbed wholly by the inner stream, but that in most cases the inner stream will be absorbed by the outer one at or below the level at which the total flow is 20 times the flow from the double orifice. There remains a strip of the (A, B)-plane adjacent to the A-axis, and there is little doubt that over most of this region double jets will experience cut-off of their inner streams. However, the cut-off levels for this region are all at very considerable distance from the source, and for most practical purposes it may be assumed that these jets retain their double character over normal working lengths.

REALITY OF THE MODEL THEORY

The turbulent exchange of fluid and of momentum between the different streams in the model described above has been assumed to follow a very simple law, according to which the rate of entrainment of adjacent fluid by a turbulent stream depends only on the velocity difference across that part of the shear flow from which is supplied the turbulent energy in that stream. This type of entrainment law has proved extremely effective in a wide range of problems involving a single column or layer of turbulent flow (particularly for jets, wakes and plumes), and is supported by dimensional arguments. It seems reasonable to assume also that it will yield information on the exchange of fluid and of momentum between adjacent turbulent streams, and so provide a simple approach to such problems as the double jet, the jet or plume in a turbulent environment and the jet or plume in turbulent pipe flow.

Before such treatments can be used with confidence, an experimental verification of the predictions of the treatment given above, or of some other case, is absolutely essential. Moreover, this test is likely to prove a good deal more difficult than the corresponding tests which have been made, for example, on buoyant plumes [3], because of the difficulty in determining the "mean position of the interface between two turbulent streams". However, the results for double jets, and it must be stressed that these are given in terms of mean velocities, mean jet radii and mean concentrations, suggest that there are quite strong differences between the inner and outer streams for suitable source conditions. Thus it should be possible to measure differences in concentration, although buoyancy effects must not be introduced without modification of the model. Measurement of the height to cut-off of the inner jet is not likely to prove rewarding, as by this level the differences between the two streams have been greatly reduced and the top of the inner jet may have little meaning from an experimental point of view.

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Résumé—Cet article présente une étude théorique de jets coaxiaux turbulents produits par l'émission, dans une même direction avec des vitesses différentes, de deux courants de fluide miscibles, à partir de deux tuyères concentriques. On considère le domaine où les deux courants peuvent se mélanger librement. Cette étude est valable dans le cas où le fluide ambiant est au repos ou en écoulement laminaire à vitesse uniforme, dans la direction du jet, mais ne tient pas compte de l'effet du mélange à travers la frontière extérieure de la couche limite provoqué par la turbulence pouvant exister dans l'écoulement principal.

La théorie est fondée sur des hypothèses simples en ce qui concerne les processus de mélange entre les courants et le fluide ambiant, elle détermine en principe la concentration des deux fluides sources dans chaque partie du double jet et l'étalement des couches limites en fonction de la distance de la source. La dernière partie de l'étude est de caractère numérique; quelques cas particuliers sont présentés et une revue générale des solutions est faite pour toutes les conditions de fonctionnement des tuyères.

Zusammenfassung—Für koachsiale turbulente Doppelstrahlen ist eine theoretische Behandlung angegeben. Derartige Strahlen entstehen beim gleichsinnigen Ausströmen zweier ineinander mischbarer Flüssigkeiten unterschiedlicher Geschwindigkeit aus einer inneren und einer konzentrischen äusseren Düse in eine umgebende Flüssigkeit, die mit jeder Strahlflüssigkeit vollkommen mischbar ist. Die Überlegungen gelten sowohl für ruhende, als auch für gleichmässig in Strahlrichtung laminar strömende Umgebungsmedien; sie schliessen jedoch den Mischeffekt an der äusseren Begrenzung infolge möglicher Turbulenz im Hauptstrom aus. Die Theorie beruht auf einfachen Modellannahmen für den Mischvorgang zwischen den Strahlen und mit der Umgebung. Im Prinzip ergibt sie die Konzentration der zwei Quellflüssigkeiten an jeder Stelle des Doppelstrahls und die Ausbreitung der Begrenzungen mit zunehmendem Abstand von der Quelle. Der Schlussteil der Lösung ist numerisch; einige repräsentative Fälle sind angegeben, ein allgemeiner Überblick zeigt den Lösungscharakter für alle Düsenbedingungen.

Аннотация—Дается теоретический метод рассмотрения двух коаксиальных турбулентных струй, образующихся при выходе двух смешивающихся потоков жидкости с разными скоростями. При этом истечение происходит в одном направлении из внутреннего и внешнего концентрических сопел в область течения жидкости, с которой оба потока свободно смешиваются. Этот метод может найти применение при неподвижной окружающей среде или при её движении в виде однородного ламинарного потока в одном направлении со струей при отсутствии смешения во внешнем пограничном слое, вызванного турбулентностью, возможной в основном потоке. Теория, основанная на простых модельных представлениях о процессах смешения потоков и окружающей среды, определяет, в основном, концентрацию двух жидких источников в обеих частях сдвоенной струи и расстояния границ при удалении от источника. В конце дается численное решение и описывается несколько характерных случаев. Дается общий обзор решений для всех условий сопла.